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Abstract

Broadcasting is a problem of information dissemination described for a group of individuals connected by a communication network, where one individual has an item of information and needs to communicate it to everyone else. Numerous previous papers have investigated ways to construct sparse undirected graphs (networks) in which this process can be completed in minimum time. In this paper, we consider the broadcast problem in directed graphs. We describe and recall some techniques to construct sparse digraphs on n vertices in which broadcasting can be completed in minimum time. For $n=2^p-1$ and $n=2^p-2$, we show that these techniques produce the sparsest possible digraphs of this type (called minimum broadcast digraphs, or *MBD*s). In the case $n=2^p-1$, we give one class of *MBD*s, as for the case $n=2^p-2$, we give two non isomorphic classes of *MBD*s. We also show a class of *MBD*s on $n=2^p$ vertices which is non isomorphic to the one given in Liestman and Peters (Discrete Appl. Math. 37/38 (1992) 401–419). For some other infinite classes of values of n ($n=2^{p-1}+1$, $n=2^p-3$ and $n=2^p-4$), we also give techniques that produce the best lower and upper bounds on the size of *MBD*s known so far.

Keywords: Broadcasting; Directed graphs; Networks

1. Introduction

Broadcasting refers to the process of dissemination of information in a communication network where a message, originated by one member, has to be transmitted to all the other members of the network. These communication patterns find their main applications in the field of interconnection networks for parallel and distributed architecture. They are achieved by placing communication calls over the communication lines of the network. We will consider a *constant-time*, *1-port* model, that is each call requires one unit of time, a vertex can participate in only one call per unit and a vertex can only call a vertex to which it is adjacent. Given a strongly connected

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digraph G, let $\vec{b}(v)$ be the time for vertex v to broadcast in G. The *broadcast time* of G, denoted by $\vec{b}(G)$, is then defined as follows: $\vec{b}(G) = \max_{v \in V(G)} \vec{b}(v)$. If we consider the complete digraph K_n^* of order n, we can easily see that $\vec{b}(K_n^*) = \lceil \log_2(n) \rceil$. Let $\vec{b}_n$ be this value of $\vec{b}(K_n^*)$. A *broadcast digraph* will denote any digraph able to broadcast in minimum time. However, it is not necessary to consider the complete digraph K_n^* to get a broadcast digraph. Any broadcast digraph with a minimum number of directed edges is then called a *Minimum Broadcast Digraph*, or *MBD*. Its number of directed edges will be denoted by $\vec{B}(n)$.

Practically, the study of Minimum Broadcast Digraphs aims to determine communication networks with a minimum number of communications links, in which broadcasting can be achieved from any vertex in minimum time.

Analogous definitions have been previously given for undirected graphs (cf. [5]): the broadcast time of a vertex v in a graph G will be denoted by $b(v)$, and the number of edges of a *Minimum Broadcast Graph*, or *MBG*, is denoted by $B(n)$.

Note that, throughout the paper, the *neighbours* of a vertex u will be the vertices v such that there is a directed edge uv (in that case, we will sometimes use the term *out-neighbours* of u), *or* a directed edge vu (in that case, we will sometimes use the term *in-neighbours* of u). Let then the *indegree* (resp. *outdegree*) of vertex u, or $d^-(u)$ (resp. $d^+(u)$), be its number of in-neighbours (resp. of out-neighbours).

This paper is organized as follows: Section 2 will recall some known general results given in [7, 8]. Section 3 will be devoted to new general results on $\vec{B}(n)$, while in Section 3.5, we give a summary of these results for n in the range 1–32.

2. Known results

In this section, we intend to recall general results about $\vec{B}(n)$ for infinite classes of values of n. Note, though, that many particular cases have been sorted out in [7], that we will not recall here. We refer to [7, 8] for a more detailed information about the structure of *MBD*s.

In [8], however, the aim of the study is not to find *MBD*s. Their goal was to find broadcast digraphs [1] that have the property of being regular. Those digraphs will consequently give us upper bounds for $\vec{B}(n)$. In particular, Park and Chwa build a class of circulant digraphs and show that they are regular broadcast digraphs.

Definition 1. A *circulant digraph* on n vertices $C_n'(a_1, a_2, \ldots, a_p)$, $a_1 < a_2 < \cdots < a_p$, has vertex set $V = \{v_0, v_1, \ldots, v_{n-1}\}$ and edge set $E = \{(v_x, v_y) \mid \exists\ a_i,\ 1 \leqslant i \leqslant p$ such that $x + a_i \equiv y \pmod n)\}$.

Park and Chwa showed that $C_n'(2^1-1, 2^2-1, \ldots, 2^{\lfloor \log_2 n \rfloor}-1)$ is a broadcast digraph for any n. Moreover, such a digraph is $\lfloor \log_2 n \rfloor$-regular. This theorem can be transformed

[1] Note that Park and Chwa [8] refer to broadcast digraphs as *minimal broadcast digraphs*.

into a general upper bound for $\vec{B}(n)$. Indeed, if n is not a power of 2, $\lfloor \log_2 n \rfloor = \vec{b}_n - 1$, where $\vec{b}_n$ is the broadcast time. Hence the following theorem.

Theorem 1. *For all* $2^{p-1} + 1 \leqslant n \leqslant 2^p - 1$, $\vec{B}(n) \leqslant n(p-1)$.

Moreover, Park and Chwa [8] showed the following theorems.

Theorem 2. *For all* $2^{p-1} + 1 \leqslant n \leqslant 2^{p-1} + 2^{p-2}$ *with* $p \geqslant 4$, *there exists a regular broadcast digraph of order n and regular of degree* $\lfloor \log_2 n \rfloor - 1$.

Theorem 3. *For all* $2^{p-1} + 1 \leqslant n \leqslant 2^{p-1} + 2^{p-4}$ *with* $p \geqslant 5$, *there exists a regular broadcast digraph of order n and regular of degree* $\lfloor \log_2 n \rfloor - 2$.

These theorems can be translated to the following ones:

Theorem 4. *For all* $2^{p-1} + 1 \leqslant n \leqslant 2^{p-1} + 2^{p-2}$ *with* $p \geqslant 4$, $\vec{B}(n) \leqslant n \cdot (p-2)$.

Theorem 5. *For all* $2^{p-1} + 1 \leqslant n \leqslant 2^{p-1} + 2^{p-4}$ *with* $p \geqslant 5$, $\vec{B}(n) \leqslant n \cdot (p-3)$.

Finally, Liestman and Peters [7] have shown the following theorem, which is the only exact known general value of $\vec{B}(n)$ for an infinite class of values of n.

Theorem 6. $\vec{B}(2^p) = p2^p$.

Indeed, it is not difficult to see that any vertex of outdegree strictly less than p cannot inform $n = 2^p$ vertices in minimum time. Moreover, we can take any (undirected) *MBG* on 2^p vertices and replace each edge with a pair of symmetric directed edges to get a broadcast digraph (a technique originally observed by [4]), hence the result. This is what was done in [7], taking H_p (the hypercube of dimension p) as *MBG*, and replacing each undirected edge with a pair of symmetric directed edges, in order to get a graph that we can call the "directed hypercube" H_p^*.

3. New results

In this section, we will give new results about Minimum Broadcast Digraphs, mainly by giving bounds on $\vec{B}(n)$ for different values of n. In particular, in order to get lower bounds on $\vec{B}(n)$, we will often use the following argument: if a vertex of outdegree d cannot inform up to n vertices in $\vec{b}_n$ time units, then every vertex in an MBD_n is of outdegree at least $d+1$. This has been discussed previously in [7] (Lemma 6); the following lemma is just another way of stating the results from that paper, which will be useful for our purpose.

Lemma 1 (Liestman and Peters [7]). *Let G be a digraph of order n, and $p=\lceil \log_2(n)\rceil$. A vertex of outdegree d in G can inform at most $2^p - 2^{p-d} + 1$ vertices in p time units.*

3.1. A new class of MBDs of order 2^p

Theorem 7. *The family of circulant digraphs $C'_{2^p}(1,3,\ldots,2^p-1)$ $(p\geqslant 3)$ is a class of MBDs on 2^p vertices non isomorphic to the class of "directed hypercubes" H_p^*.*

Proof. First, it is not difficult to see that $C'_{2^p}(1,3,\ldots,2^p-1)$ is an *MBD* for $n=2^p$, since in that case $\lfloor \log_2 n\rfloor = \lceil \log_2 n\rceil = p$, and consequently such a digraph is of minimum size for broadcasting.

Moreover, we know that H_p^* is also an *MBD* for any p (cf. Theorem 6 above). Note that H_p^* is such that each of its vertices u has p neighbours, and for each v neighbour of u, there is a directed edge uv and a directed edge vu.

Now let us show that $C'_{2^p}(1,3,\ldots,2^p-1)$ and H_p^* are nonisomorphic for any $p\geqslant 3$. Suppose they are isomorphic. In that case, each vertex u of $C'_{2^p}(1,3,\ldots,2^p-1)$ has p neighbours, and for each of these neighbours v, there is a directed edge uv and a directed edge vu. Now let $u=v_0$ and $v=v_3$. By definition of $C'_{2^p}(1,3,\ldots,2^p-1)$, there would be a k such that $3+2^k-1\equiv 0 \bmod n$, that is $2^k+2=2^p$. This is only possible for $p=2$ and $k=1$. Consequently, for any $p\geqslant 3$, $C'_{2^p}(1,3,\ldots,2^p-1)$ and H_p^* are nonisomorphic. □

3.2. Exact values of $\vec{B}(2^p-1)$ and $\vec{B}(2^p-2)$

Proposition 1. *For all $p\geqslant 3$:*
- $\vec{B}(2^p-2)=(p-1)(2^p-2)$;
- $\vec{B}(2^p-1)=(p-1)(2^p-1)$.

Proof. In both cases, that is $n=2^p-1$ and $n=2^p-2$, it is not difficult to see that any vertex of outdegree strictly less than $(p-1)$ cannot inform more than 2^p-3 vertices within p time units (by Lemma 1). Hence $\vec{B}(n)\geqslant n(p-1)$. Moreover, it follows from the result given by Park and Chwa (cf. Theorem 1) that $\vec{B}(n)\leqslant n\cdot(p-1)$; hence the result. Consequently, for any $n=2^p-1$ or $n=2^p-2$, $C'_n(1,3,7,\ldots,2^{\lfloor \log_2 n\rfloor}-1)$ is an *MBD*. □

3.3. A second class of MBDs for $n=2^p-2$

We have seen that the circulant digraphs $C'_n(1,3,\ldots,2^{\lfloor \log_2 n\rfloor}-1)$ were *MBD*s for $n=2^p-1$ and $n=2^p-2$. However, there is a second class of *MBD*s for $n=2^p-2$ which is nonisomorphic to the circulant digraphs defined above for any $p\geqslant 3$. They are what we can call the *Knödel digraphs*. Below, is a definition of the Knödel graphs in the undirected case.

Definition 2. The *Knödel graph* [6, 3] on $n \geqslant 2$ vertices (n even) and of maximum degree $1 \leqslant \Delta \leqslant \lfloor \log_2(n) \rfloor$ is denoted $W_{\Delta,n}$. The vertices of $W_{\Delta,n}$ are the couples (i, j) with $i = 1, 2$ and $0 \leqslant j \leqslant (n/2) - 1$. For every j, $0 \leqslant j \leqslant (n/2) - 1$, there is an edge between vertex $(1, j)$ and every vertex $(2, j + 2^k - 1 \bmod (n/2)$, for $k = 0, \ldots, \Delta - 1$.

For $0 \leqslant k \leqslant \Delta - 1$, an edge of $W_{\Delta,n}$ which connects a vertex $(1, j)$ to the vertex $(2, j + 2^k - 1 \bmod n/2)$ is said to be *in dimension k*.

It has been shown in [2] that $W_{p-1,n}$ is a gossip graph (hence a broadcast graph) for any even n not a power of 2 and $p = \lceil \log_2 n \rceil$. It suffices for any vertex u to communicate at time $1 \leqslant t \leqslant p - 1$ along dimension $(t - 1)$, and, during the last time unit, to communicate again along dimension 0.

Now let a *Knödel digraph* $W^*_{\Delta,n}$ be a Knödel graph where each undirected edge is replaced by a symmetric pair of directed edges. In that case, it is easy to see that $W^*_{p-1,n}$ is a broadcast digraph of size $n \cdot (p - 1)$ for any even n not a power of 2. Hence, in the case $n = 2^p - 2$, the Knödel digraph $W^*_{p-1,n}$ is an *MBD*.

Theorem 8. *$W^*_{p-1,n}$ and $C'_n(1, 3, \ldots 2^{p-1} - 1)$ are two nonisomorphic classes of MBDs of order $n = 2^p - 2$ for $p \geqslant 3$.*

Proof. Suppose $n = 2^p - 2$, and let us look at the number of neighbours of any vertex u in each digraph. By definition, in $W^*_{p-1,n}$, a vertex u has $(p - 1)$ neighbours v_i, with, for each of them, a directed edge uv_i and a directed edge $v_i u$. Note also that, by definition, in $C'_n(1, 3, \ldots, 2^{p-1} - 1)$, each vertex has at least $(p - 1)$ neighbours.

Now suppose $W^*_{p-1,n}$ and $C'_n(1, 3, \ldots, 2^{p-1} - 1)$ are isomorphic. In that case, in $C'_n(1, 3, \ldots, 2^{p-1} - 1)$, every vertex u would have exactly $(p - 1)$ neighbours, and for every neighbour v of u, there is a directed edge uv and a directed edge vu. Let $u = v_0$ and $v = v_{2^p-3}$. They are neighbours in $C'_n(1, 3, \ldots, 2^{p-1} - 1)$ by definition. Provided that the two digraphs are isomorphic, we know that there is a directed edge $v_{2^p-3}v_0$ and a directed edge $v_0 v_{2^p-3}$. By definition of $C'_n(1, 3, \ldots 2^{p-1} - 1)$, the only possible case is when $p = 2$. Hence, $W^*_{p-1,n}$ and $C'_n(1, 3, \ldots 2^{p-1} - 1)$ are two nonisomorphic classes of *MBD*s for $p \geqslant 3$. □

3.4. Bounds for $\vec{B}(n)$

Before giving new bounds for $\vec{B}(n)$, it is necessary to define here the notion of *minimum broadcast tree*. Let u be a vertex in a digraph G of order n, with $d^+(u) = k$. We would like to know if u is able to broadcast its information in G in minimum time. The method here is to build the tree which represents the best information dissemination from u to all the other vertices in G, respecting the constraint on u's outdegree. If this tree contains strictly less than n vertices, we know that u will not be able to broadcast its information in minimum time, hence G is not a broadcast digraph.

We then call *minimum broadcast tree* rooted at u the tree representing the best information dissemination which can occur from u. Such an example of a *minimum broadcast tree* is given in Fig. 1, where $d^+(u) = 1$ and $n = 17$.

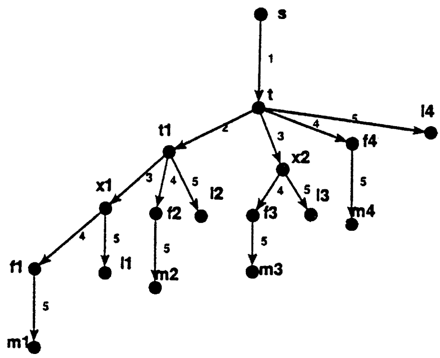

Fig. 1. Minimum broadcast tree rooted at s, a vertex of outdegree 1.

Theorem 9. $7\times 2^{p-3}+1\leqslant \vec{B}(2^{p-1}+1)\leqslant 9\times 2^{p-3}-2$ *for any* $p\geqslant 4$.

Proof. Let us prove first the lower bound. When $n=2^{p-1}+1$, there can be vertices of outdegree 1 in an MBD_n, and in that case such a vertex, say s, can inform at most n vertices within p time units (by Lemma 1). Note that if there is no vertex of outdegree 1 in an MBD_n, then it follows directly that $\vec{B}(n)\geqslant 2n$. Fig. 1 shows the minimum broadcast tree rooted at s in the case $n=17$, which will help to illustrate the general proof.

Let $n=2^{p-1}+1$ and let s be a vertex of outdegree 1 in an MBD_n. Then, as shown in Fig. 1, s can inform at most n vertices. In that case, it is not difficult to see that, in the minimum broadcast tree rooted at s, say T, there is 1 vertex t of outdegree $p-1$, 1 vertex t_1 of outdegree $p-2$, 2 vertices x_1 and x_2 of outdegree $p-3$, 4 vertices of outdegree $p-4,\ldots,2^{p-4}$ vertices of outdegree 2. Apart from those vertices, there remains $n_1=3\times 2^{p-3}$ vertices in the tree, for which their outdegree is at least 1 in the MBD_n. Among those n_1 vertices, there are 2^{p-3} leaves m_i such that their father is of outdegree at least 2 in the tree, and 2^{p-3} leaves m_i such that their father is of outdegree 1 in the tree. Let us focus on that last class of leaves. Let m be such a leaf, and f its father in the tree. If both are of outdegree 1 in the MBD_n, the minimum broadcast tree rooted at f would hold strictly less than n vertices. Hence, $d^+(f)+d^+(m)\geqslant 3$. Now if we compute the sum S of all the vertices outdegrees, we get $S\geqslant 1+(p-1)+(p-2)+2(p-3)+4(p-4)+\cdots+2^{p-4}\times 2+2^{p-3}+3\times 2^{p-3}$, that is $S\geqslant 7\times 2^{p-3}$. Since $\vec{B}(n)\geqslant S$, we get $\vec{B}(n)\geqslant 7\times 2^{p-3}$. Note that this lower bound on $\vec{B}(n)$ can be generalized for all $n=2^p-2^{p-d}+1$ with $p\geqslant 2d+1$. This will be the purpose of Theorem 10.

Now suppose $\vec{B}(n)=7\times 2^{p-3}$. Then the only configuration for the MBD_n is $d^+(l)=1$ for each leaf l of the tree, and $d^+(f)=2$ for each vertex f such that it was of outdegree 1 in the tree (indeed, if for a vertex f_i we have $d^+(f_i)=1$ in an MBD_n, then we necessarily have $d^+(l_i)\geqslant p-1\geqslant 3$, which contradicts the hypothesis on $\vec{B}(n)$). All the other vertices have the same outdegree in the MBD_n than in the minimum broadcast tree T.

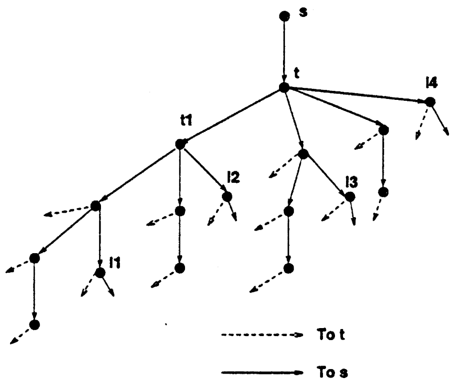

Fig. 2. A broadcast digraph on 17 vertices.

Let t be the out-neighbour of s. Since t is the only vertex of outdegree $p-1$ in the *MBD*, it must be neighbour of all the leaves l. Then each directed edge lx will be an edge lt. Now, there remains to add one directed edge fx for every f. Necessarily, at least one of these edges must be f_is, otherwise no vertex could inform s. Let s be in-neighbour of f_k. In that case, the minimum broadcast tree rooted at f_k holds strictly less than n vertices. Hence $\vec{B}(n) \geqslant 1 + 7 \times 2^{p-3}$.

The upper bound derives from the following construction: let s be a vertex of outdegree 1, and let us build a minimum broadcast tree rooted at s, say T. Let t be the son of s in T. Note that, by construction, t is of outdegree $p-1$ in T. Let t_1 be the son of t such that $d^+(t_1) = p-2$ in T, and let l_i be the leaves of the tree such that their father is of outdegree at least 2 in T. To the minimum broadcast tree rooted at s we add all the directed edges v_it for every vertex $v_i \notin \{s, t, t_1\}$, and all the directed edges l_is for all i. An example of this construction is given in Fig. 2 where $n = 17$.

Now let us prove that the digraph constructed as above is a broadcast digraph and holds $9 \times 2^{p-3} - 2$ edges.

The minimum broadcast tree has $(n-1)$ edges. We add $(n-3)$ edges of the form v_it and 2^{p-3} edges of the form l_is. Hence, the number of edges is $2^{p-1} + 2^{p-1} - 2 + 2^{p-3}$, that is $9 \times 2^{p-3} - 2$.

Let us now prove that this construction gives broadcast digraphs. Let T be the minimum broadcast tree rooted at s which is clearly visible in Fig. 2. First, it is easy to see that for vertices s and t, broadcast can be made in minimum time to all the vertices of the digraph. t_1 can also broadcast its piece of information in minimum time as follows: it informs $l_{2^{p-4}}$ (l_2 in Fig. 2) during the first round. $l_{2^{p-4}}$ then informs t at round 2, and s at round 3. Then t and t_1 can inform all the other vertices of the digraph within $p-1$ rounds: t_1 will begin at round 2 and broadcasts as in T (except for vertex $l_{2^{p-4}}$), while t begins at round 3 and broadcasts as in T as well.

For all the leaves l_i, it is not difficult to see that broadcast can be made in minimum time too: let l_i inform t during the first time unit; t will then broadcast the information

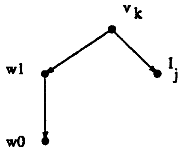

Fig. 3. Subtree of T rooted at v_k, vertex of outdegree 2 in T.

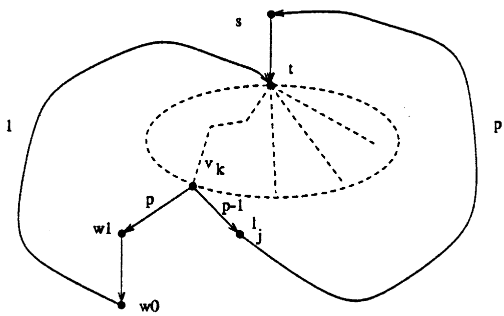

Fig. 4. A broadcast scheme for a vertex $w0$.

to the rest of the vertices, except s and l_i, the same way as in T. Then s can be informed by l_i during time unit 2, for instance.

It remains to prove that every other vertex v_i can broadcast in this digraph in minimum time. Let us distinguish two classes of vertices. First, consider the vertices v_i such that they are of outdegree at least 2 in T. Hence, the subtree of T rooted at v_i, say T_{v_i}, holds at least one leaf l_j. Let v_i inform t at time unit 1: t will then broadcast v_i's information to $T - \{T_{v_i} \cup s\}$ as it did in T. Now v_i still needs to inform $T_{v_i} \cup s$. Recall that in T, v_i could not inform the vertices of T_{v_i} before time unit 3. If v_i informs now the vertices of T_{v_i} from time unit 2, this means that l_j will be informed before the last time unit. Then l_j can inform s during the last time unit, p, hence v_i has broadcast its information to all the vertices of the digraph.

Now let us consider the vertices w_i of outdegree less than or equal to 1 in T, and let us distinguish two cases: either they are of outdegree 1 in T (let us call those vertices $w1$), or they are of outdegree 0 in T (let us call them $w0$). Fig. 3 shows the subtree of T rooted at v_k, father of a $w1$ in T. Note that the other son of v_k is a leaf l_j, as v_k is of outdegree 2 in T.

Let us distinguish the two classes of vertices w_0 and w_1:

- Let $w0$ inform t at time unit 1. Then t can inform $T - \{w0, s\}$ as it did in T. However, if we swap time units $p-1$ and p during which v_k communicated with, respectively, $w1$ and l_j in T, and if l_j informs s during time unit p, then $w0$ has informed all the vertices of the digraph in minimum time. We refer to Fig. 4 for a better understanding of the method.
- Analogously, let $w1$ inform t during time unit 1 and let T inform $T - \{w0, w1, s\}$ as it did in T, except for l_j which will be informed at time unit $p-1$ instead of

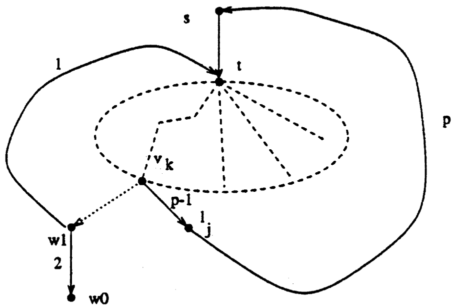

Fig. 5. A broadcast scheme for a vertex $w1$.

p. Then l_j can inform s at time unit p, and $w1$ can inform $w0$ at time unit, say, 2. Fig. 5 shows this broadcast scheme. Hence $w1$ can broadcast its information to all the vertices in minimum time.

Every vertex of the digraph can broadcast its information to all the vertices in minimum time. Hence, the general construction always give broadcast digraphs, and $\vec{B}(n) \leqslant 9 \times 2^{p-3} - 2$. □

Note that the general upper bound given in this theorem matches the upper bounds given in [7] for $n = 9$ and 17.

As mentioned in proof of Theorem 9, it is possible to generalize the result on the lower bound for $\vec{B}(n)$ when $n = 2^p - 2^{p-d} + 1$ and $p \geqslant 2d + 1$. Indeed, we have the following theorem.

Theorem 10. *For all* $n = 2^p - 2^{p-d} + 1$ *with* $p \geqslant 2d + 1 \geqslant 3$, $\vec{B}(n) \geqslant d(n-1) + 2^{p-1} - 2^{p-2d-1}$.

Proof. First, we know by Lemma 1 that for such values of n, there cannot be any vertex of outdegree less than or equal to $d - 1$ in an MBD_n. Moreover, if there is a vertex u of outdegree d in an MBD_n, then it can inform at most n vertices in $\lceil \log_2(n) \rceil = p$ time units. In that case, the minimum broadcast tree rooted at u holds exactly n vertices. Since we suppose $p \geqslant 2d+1$, we also know that any son of u in this tree is of outdegree at least $d + 1$ (constraint (C1)). We refer to Fig. 6 for an example of a minimum broadcast tree rooted at vertex u, where $n = 25$, $d = 2$ and $p = 5$.

Now let us suppose that there exists a vertex u of outdegree d in an MBD_n. Note first that if this is not the case, we immediately have $\vec{B}(n) \geqslant n(d + 1)$, and standard calculations then show the correctness of the theorem. If there exists such a vertex u in the digraph, let us construct T, the minimum broadcast tree rooted at u, and let us focus on $d + 1$ different types of subtrees of T. These types of subtrees are denoted T_k, where $1 \leqslant k \leqslant d + 1$. They are defined as follows:

- T_{d+1} is a branch of T rooted at a vertex which receives the message from u at round $p - d$;

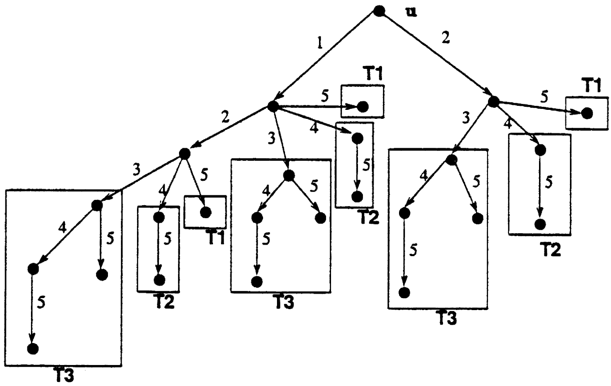

Fig. 6. Example of a minimum broadcast tree rooted at u of outdegree 2, where $n=25$.

- T_d is a branch of T rooted at a vertex which receives the message from u at round $p-d+1$, not being part of any T_{d+1};
- ...
- T_1 is a branch of T rooted at a vertex which receives the message from u at round p, not being part of any T_i with $i \geqslant 2$.

Generally speaking, the T_k are defined in a descending way, from T_{d+1} to T_1, where T_k (with $1 \leqslant k \leqslant d+1$) is a branch of T rooted at a vertex which receives the message from u at round $p-k+1$, not being part of any $T_{k'}$ with $k' > k$. We refer to Fig. 6 for an example: notably, we see that T_1 consists of a single vertex, and T_2 consists of two vertices joined by a directed edge.

It is not difficult to see that, for any fixed k, $1 \leqslant k \leqslant d+1$, there are exactly $A=(2^d-1)2^{p-2d-1}$ subtrees of T which are of type T_k.

Now, the method consists in counting, for each k, the sum S_k of the outdegrees (in the MBD_n) over all the vertices of a T_k. First, we know by Lemma 1 that $S_1 \geqslant d$. Now let us prove that $S_k \geqslant 2^{k-2}(2d+1)$ for any $2 \leqslant k \leqslant d+1$. Indeed, by definition, each T_k, $3 \leqslant k \leqslant d+1$, can be seen as 2 copies of a subtree of type T_{k-1} joined by a directed edge vw, where v sends its information to w during round $p-k+2$ (cf. Fig. 7). In that case, let us consider the subtree of type S_2 and proceed by induction for any $k \geqslant 3$. A subtree of type T_2 consists of two vertices, v and w, joined by a single directed edge vw. Let us now discuss v and w's outdegrees in the MBD_n. Suppose first that $d^+(v)=d$; in that case, we know by constraint C1 that $d^+(w) \geqslant d+1$. And if $d^+(v) \geqslant d+1$, we know by Lemma 1 that $d^+(w) \geqslant d$. On the whole, we get $S_2 \geqslant 2d+1$. Owing to the recursive construction of T_k mentioned above for any $k \geqslant 3$, the induction applies directly and we get the following result: for any $2 \leqslant k \leqslant d+1$, $S_k \geqslant 2^{k-2}(2d+1)$.

Owing to the above study, we can now compute the number of directed edges which are necessary for the digraph to be an MBD_n; this will give a lower bound on $\vec{B}(n)$. Let N be the number of directed edges in T ($N=n-1$), and N_k be the number of directed edges of a subtree of type T_k, for $1 \leqslant k \leqslant d+1$. We have: $\vec{B}(n) \geqslant N + A(\sum_{k=1}^{d+1} S_k - \sum_{k=1}^{d+1} N_k)$. It is not difficult to see that $N_k = 2^{k-1}-1$ for any $1 \leqslant k \leqslant d+1$. Hence we

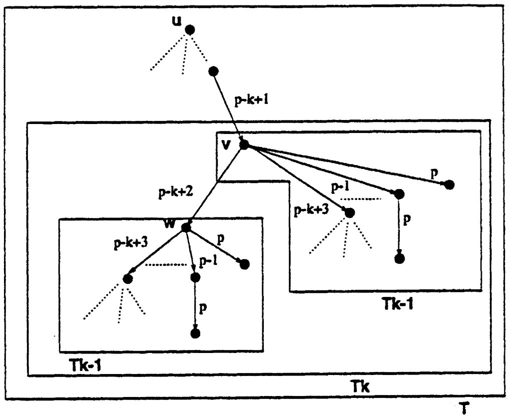

Fig. 7. T_k can be constructed from two copies of a T_{k-1}.

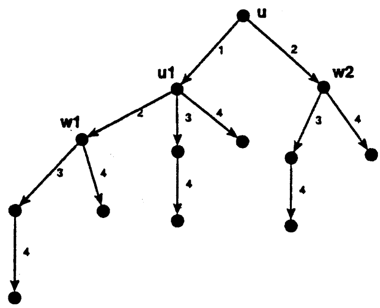

Fig. 8. Minimum broadcast tree on 13 vertices.

have $\sum_{k=1}^{d+1} S_k - \sum_{k=1}^{d+1} N_k = d + (2^d - 1)(2d+1) - (2^{d+1} - d - 2) = 2^d(2d-1)+1$. Finally, we get $\vec{B}(n) \geqslant (n-1) + (2^d - 1)2^{p-2d-1}(2^d(2d-1)+1)$, and standard calculations give us the final result. $\square$

Theorem 11. *For all* $n = 2^p - 3$ *with* $p \geqslant 4$, $n(p-2) + 3 \leqslant \vec{B}(n) \leqslant n(p-1) - 1$.

Proof. In [7], Liestman and Peters gave an equivalent of Farley's two-way split method for broadcast digraphs. This method gives the following formula: $\vec{B}(n) \leqslant \vec{B}(n_1) + \vec{B}(n_2) + 2n_2$, where $n_1 + n_2 = n \geqslant 4$, $n_1 \geqslant n_2$ and $\lceil \log_2 n_1 \rceil = \lceil \log_2 n_2 \rceil = \lceil \log_2 n \rceil - 1$. Using this method, we get the upper bound on $\vec{B}(2^p - 3)$ where $n_1 = 2^{p-1} - 1$ and $n_2 = 2^{p-1} - 2$.

If we have a vertex u of outdegree $(p-2)$ in an MBD_n, it will be able to inform exactly $n = 2^p - 3$ vertices within p time units (by Lemma 1), as shown in Fig. 8 for the case $n = 13$. But this implies that the vertex informed by u after the first time unit, say u_1, is of outdegree $(p-1)$ at least in the *MBD*. In the broadcast tree T rooted at u, there are two other vertices w_1 and w_2 which are of outdegree at least $(p-2)$. W.l.o.g., let us consider w_1 and its outdegree in the *MBD*: either w_1 is of outdegree at least

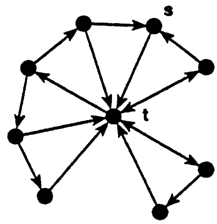

Fig. 9. A *MBD* on 9 vertices.

$(p-1)$, or it is of outdegree $(p-2)$ and then one of its out-neighbours (necessarily one of its sons in the tree) is of outdegree at least $(p-1)$. In either case, at least three vertices in the digraph are of outdegree at least $(p-1)$, hence the result. □

Theorem 12. *For all* $n=2^p-4$ *with* $p\geqslant 4$, $n(p-2)\leqslant \vec{B}(n)\leqslant n(p-\frac{3}{2})$.

Proof. Any vertex of outdegree strictly less than $(p-2)$ in a MBD_n can inform up to 2^p-7 vertices (cf. Lemma 1), hence the lower bound. The upper bound derives from an upper bound given in [1] in the undirected case. Indeed, it has been shown that $B(2^p-4)\leqslant \frac{n}{2}(p-\frac{3}{2})$. As $\vec{B}(n)\leqslant 2B(n)$ for any n [4], we get the result. □

Note that it would be possible to go on for $n=2^p-5$, $n=2^p-6$, etc. However, for $n=2^p-3$ and 2^p-4, the bounds presented above give new results in the range 1–32 (namely, $n=28$ and 29), while this is not the case for $n\leqslant 2^p-5$.

3.5. Summary of the results for n in the range 1–32

The table displayed in Fig. 10 shows, respectively, lower and upper bounds for $\vec{B}(n)$ for n in the range 1–32. The asterisk indicates optimality, and bounds printed in bold characters indicate new results.

All these bounds come from the results given in this paper, except the two new upper bounds for $\vec{B}(n)$ with $n=27$ and 29, which derive from the fact that $B(\vec{n})\leqslant 2B(n)$ for any n [4], and that $B(27)=44$ and $B(29)=52$ by Saclé [9].

Finally, note that these results give, for some small values of n, MBD_n which are nonisomorphic to the ones given in [7]. In particular, we get the following proposition.

Proposition 2. *The digraph shown in Fig.* 9 *is an MBD of order* 9 *nonisomorphic to the one given in* [7].

Proof. Liestman and Peters [7] proved that $\vec{B}(9)=16$ and gave one *MBD* on 9 vertices. The construction provided in proof of Theorem 9 gives broadcast digraphs with $2^{p-1}+1$ vertices and $9\times 2^{p-3}-2$ edges. Hence, in the case $p=4$, this construction gives an *MBD* on 9 vertices (cf. Fig. 9). Moreover, it is not isomorphic to the *MBD* presented in [7]: in our case, vertex t is of indegree 7 while no vertex is of indegree more than 6 in the *MBD* presented in [7]. □

n	Lower	Upper	n	Lower	Upper	n	Lower	Upper	n	Lower	Upper
1	0	0*	9	16	16*	17	**29**	34	25	**63**	75
2	2	2*	10	20	20*	18	36	36*	26	78	78*
3	3	3*	11	22	22*	19	38	39	27	81	**88**
4	8	8*	12	24	24*	20	40	40*	28	84	**96**
5	7	7*	13	29	33	21	43	53	29	**90**	**104**
6	12	12*	14	42	42*	22	45	55	30	120	120*
7	14	14*	15	45	45*	23	47	64	31	**124**	**124***
8	24	24*	16	64	64*	24	49	66	32	160	160*

Fig. 10. Summary of known results for $1 \leqslant n \leqslant 32$.

4. Conclusions and open problems

In this paper, we have studied the structure of Minimum Broadcast Digraphs in order to get better lower and upper bounds for $\vec{B}(n)$, for infinite classes of values of n. Owing to this method, together with some previous results by Park and Chwa [8], it has been possible to determine exactly $\vec{B}(2^p-1)$ and $\vec{B}(2^p-2)$. Moreover, some other general lower bounds have been given, which either match or improve the previous known results.

Owing to the construction provided by Park and Chwa [8] it has been possible to determine $\vec{B}(n)$ for $n=2^p-1$ and $n=2^p-2$. This has been made possible because any vertex of outdegree strictly less than $(p-1)$ can inform at most 2^p-3 vertices, and because a $(p-1)$-regular digraph can inform up to 2^p-1 vertices. A possible extension of this work could be to go further in this study, since any vertex of outdegree strictly less than $(p-2)$ can inform at most 2^p-7 vertices, and since a $(p-2)$-regular digraph could inform up to 2^p-4 vertices. Hence, if we manage to find a class of $(p-2)$-regular digraphs that are broadcast digraphs for any $2^p-6 \leqslant n \leqslant 2^p-4$, we would get the exact values of $\vec{B}(n)$ in that range. Note that it is true for some small values of n, such as $n=10, 11, 12$ and 26. Analogously, with a $(p-3)$-regular broadcast digraph, we could determine $\vec{B}(n)$ for $2^p-14 \leqslant n \leqslant 2^p-12$, as it is the case for $n=18$ and $n=20$. However, this method could not go further, since a $(p-4)$-regular digraph could inform at most 2^p-32 vertices, while a vertex of outdegree $(p-5)$ can inform up to 2^p-29 vertices.

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